The Hochschild-Serre property for some p-adic analytic group actions

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Abstract

Let $H \subseteq G$ be an inclusion of p-adic Lie groups. When H is normal or even subnormal in G, the Hochschild-Serre spectral sequence implies that any continuous G-module whose H-cohomology vanishes in all degrees also has vanishing G-cohomology. With an eye towards applications in p-adic Hodge theory, we extend this to some cases where H is not subnormal, assuming that the G-action is analytic in the sense of Lazard.

1 Introduction

Let $H \subseteq G$ be an inclusion of groups and let M be a G-module. If H is normal, then the *Hochschild-Serre spectral sequence* [5] has the form

$$E_2^{p,q} = H^p(G/H, H^q(H, M)) \Longrightarrow H^{p+q}(G, M). \tag{1.0.1}$$

(This is sometimes also called the *Lyndon spectral sequence* in recognition of a similar prior result [9] which did not explicitly exhibit the spectral sequence.) If H is not normal, one can still ask to what extent the G-cohomology of M is determined by the H-cohomology. In particular, one can ask whether for any morphism $M \to N$ of G-modules such that $H^i(H, M) \to H^i(H, N)$ is an isomorphism for all $i \geq 0$, $H^i(G, M) \to H^i(G, N)$ is also an isomorphism for all $i \geq 0$; in this case, we say that the inclusion $H \subseteq G$ of groups has the *HS (Hochschild-Serre) property*. Thanks to (1.0.1), the HS property holds when H is *subnormal* in G, i.e., there exists a finite sequence $H = H_0 \subset H_1 \subset \cdots \subset H_m = G$ in which each inclusion $H_i \subset H_{i+1}$ is normal. On the other hand, it is not difficult to produce examples of inclusions of finite groups for which the HS property fails; see for instance Example 2.7.

One can also exhibit an analogue of the Hochschild-Serre spectral sequence for normal inclusions of topological groups, which again implies the HS property for subnormal inclusions; see [4]. The main result of this paper (Theorem 4.1) is a restricted analogue of the HS property for certain non-subnormal inclusions of p-adic Lie groups, which applies only to the category of topological modules which are of characteristic p and *analytic* in the sense of

Lazard [8]. It is crucial that the cohomology groups of such modules can be computed using either continuous or analytic cochains; this makes it possible to quantify the statement that an analytic group action of a p-adic Lie group is "approximately abelian."

We illustrate this theorem with some examples which arise from p-adic Hodge theory. To be precise, these examples come from upcoming joint work with Liu [7] on generalizations of the Cherbonnier-Colmez theorem on descent of (φ, Γ)-modules [3], in the style of our new approach to the original theorem of Cherbonnier and Colmez [6].

2 The HS property for discrete groups

For context, we begin with some remarks on the HS property for discrete groups.

Definition 2.1. For G a group and M a G-module, we say that M has *totally trivial G-cohomology* if $H^i(G, M) = 0$ for all $i \geq 0$. Note that for given $G, H, \mathcal{C}$, the HS property can be formulated as the statement that any $M \in \mathcal{C}$ with totally trivial H-cohomology also has totally trivial G-cohomology.

Remark 2.2. If $H \subset G$ is a proper inclusion of groups and M is a G-module with totally trivial G-cohomology, M need not have totally trivial H-cohomology.

Proposition 2.3. *Let G be a finite p-group and let M be a G-module. The following conditions are equivalent.*

(a) The G-module M has totally trivial G-cohomology.

(b) The group M is uniquely p-divisible (i.e., is a module over $\mathbb{Z}[p^{-1}]$) and $H^0(G, M) = 0$.

Proof. For $i > 0$, $H^i(G, M)$ is a torsion group killed by the order of G [10, §2.4, Proposition 9]; hence (b) implies (a). Conversely, the p-torsion subgroup $M[p]$ of M has the property that $H^0(G, M[p]) = M[p]$ injects into $H^0(G, M)$. Consequently, if M has totally trivial G-cohomology, then on one hand multiplication by p is injective on M; on the other hand, the same is then true for pM (which is isomorphic to M as a G-module) and M/pM (by the long exact sequence in cohomology), but the latter forces $M/pM = 0$. Hence (a) implies (b). $\square$

Remark 2.4. Proposition 2.3 implies the HS property for inclusions of finite p-groups, although this is already clear because such inclusions are always subnormal. An immediate corollary is that if H is a subgroup of a normal subgroup P of G which is a finite p-group, then $H \subseteq G$ has the HS property.

Example 2.5 (Serre)**.** For G a semisimple algebraic group over $\mathbb{F}_q$ and P a p-Sylow subgroup, the Steinberg representation of G restricts to a free $\mathbb{F}_q[P]$-module and thus has totally trivial G-cohomology.

Here are some examples to show that the HS property does not always hold. We start with a minimal example.

Example 2.6 (Naumann)**.** Put $G = S_3$, let H be the subgroup generated by a transposition, and take $M = \mathbb{F}_3$ with the action of G being given by the sign character. It is apparent that M has vanishing H-cohomology. On the other hand, the groups $H^i(A_3, M)$ are all $\mathbb{F}_3$-vector spaces and are hence H-acyclic, so (1.0.1) yields $H^1(S_3, M) = H^1(A_3, M) = \mathbb{F}_3$. Explicitly, a nonzero class is represented by the crossed homomorphism taking one element of order 3 to $+1$ and the other to -1, mapping the other elements to 0.

A similar example exists in any odd characteristic p using the dihedral group of order $2p$. For an example in characteristic 2, we offer the following.

Example 2.7 (Serre)**.** Let M' be a 5-dimensional vector space over $\mathbb{F}_2$ equipped with a nondegenerate quadratic form q. The associated bilinear form b has rank 4; let K be its kernel and put $M = M'/K$. The action of $G = \mathrm{SO}(M', q)$ ($\cong S_6$) preserves K and the induced action on M defines an isomorphism $\mathrm{SO}(M', q) \cong \mathrm{Sp}(M, b) \cong \mathrm{Sp}_4(\mathbb{F}_2)$. The exact sequence

$$0 \to K \to M' \to M \to 0$$

of G-modules does not split, so $H^1(G, M)$ is nonzero.

Now split M as a direct sum $M_1 \oplus M_2$ of nonisotropic subspaces and put $H_i = \mathrm{SL}(M_i)$ and $H = H_1 \times H_2$ ($\cong S_3 \times S_3$). As in Example 2.5, M_1 has no nonzero H_1-invariants and restricts to a free module over $\mathbb{F}_2[P_1]$ for P_1 a 2-Sylow subgroup of H_1; it follows that M_1 has totally trivial H_1-cohomology, hence also totally trivial H-cohomology by (1.0.1). Similarly, M_2 has totally trivial H-cohomology, as then does M. We conclude that the inclusion $H \subseteq G$ does not have the HS property.

3 Analytic group actions

We now introduce the class of group actions to which our main result applies. The basic setup is taken from the work of Lazard [8].

Hypothesis 3.1. Throughout §3, let Γ be a *profinite p-analytic group* in the sense of [8, III.3.2.2]. For example, we may take Γ to be a compact p-adic Lie group.

Definition 3.2. For M a Γ-module, let $C^\cdot(\Gamma, M)$ be the complex of inhomogeneous cochains, so that $C^n(\Gamma, M) = \mathrm{Map}(\Gamma^n, M)$ and for $h \in C^n(\Gamma, M)$ and $\gamma_0, \dots, \gamma_n \in \Gamma$,

$$\begin{aligned}(dh)(\gamma_0, \dots, \gamma_n) = \gamma_0 h(\gamma_1, \dots, \gamma_n) \\ &+ \sum_{i=1}^{n} (-1)^i h(\gamma_0, \dots, \gamma_{i-2}, \gamma_{i-1}\gamma_i, \gamma_{i+1}, \dots, \gamma_n) \\ &+ (-1)^{n+1} h(\gamma_0, \dots, \gamma_{n-1}).\end{aligned}$$

For M a topological Γ-module, let $C^\cdot_{\mathrm{cont}}(\Gamma, M)$ be the subcomplex of $C^\cdot(\Gamma, M)$ consisting of continuous cochains, so that $C^n_{\mathrm{cont}}(\Gamma, M) = \mathrm{Cont}(\Gamma^n, M)$. Let $H^\cdot_{\mathrm{cont}}(\Gamma, M)$ be the cohomology groups of $C^\cdot_{\mathrm{cont}}(\Gamma, M)$, topologized as subquotients for the compact-open topology; for a more intrinsic interpretation of these groups, see [4, Proposition 9.4].

For normal subgroups of Γ, we again have a Hochschild-Serre spectral sequence.

Lemma 3.3. *For any closed normal subgroup Γ' of Γ and any topological Γ-module M, there is a spectral sequence*

$$E_2^{p,q} = H^p_{\mathrm{cont}}(\Gamma/\Gamma', H^q_{\mathrm{cont}}(\Gamma', M)) \Longrightarrow H^{p+q}_{\mathrm{cont}}(\Gamma, M).$$

For our purposes, convergence of the spectral sequence may be interpreted at the level of bare abelian groups, but it also makes sense at the level of topological groups: starting from E_2, each stage of the spectral sequence induces a subquotient topology on the subsequent stage, and $H^{p+q}_{\mathrm{cont}}(\Gamma, M)$ admits a filtration by subgroups (not guaranteed to be closed) whose subquotients are homeomorphic to the corresponding terms of E_∞.

Proof. Since Γ and Γ' are profinite, the surjection of topological spaces $\Gamma \to \Gamma/\Gamma'$ admits a continuous section. Consequently, the explicit construction of the spectral sequence for finite groups given in [5, §2] carries over without change. For further discussion, see [8, §V.3.2]. $\square$

Definition 3.4. Let A be the completion of the group ring $\mathbb{Z}_p[\Gamma]$ with respect to the p-augmentation ideal $\ker(\mathbb{Z}_p[\Gamma] \to \mathbb{F}_p)$. Put $I = \ker(A \to \mathbb{F}_p)$; we view A as a filtered ring using the I-adic filtration. We also define the associated valuation: for $x \in A$, let $w(A; x)$ be the supremum of those nonnegative integers i for which $x \in I^i$.

Definition 3.5. An *analytic* Γ*-module* is a left A-module M complete with respect to a valuation $w(M; \bullet)$ for which there exist $a > 0, c \in \mathbb{R}$ such that

$$w(M; xy) \geq a w(A; x) + w(M; y) + c \qquad (x \in A, y \in M).$$

Equivalently, there exist an open subgroup Γ_0 of Γ and a constant $a > 0$ such that

$$w(M; (\gamma - 1)y) \geq w(M; y) + a \qquad (\gamma \in \Gamma_0, y \in M).$$

Example 3.6. Let M be a torsion-free $\mathbb{Z}_p$-module of finite rank on which Γ acts continuously. Then M is an analytic A-module for the valuation defined by any basis; see [8, Proposition V.2.3.6.1].

Definition 3.7. Let M be a continuous Γ-module. A cochain $\Gamma^i \to M$ is *analytic* if for every homeomorphism between an open subspace U of Γ^i and an open subspace V of $\mathbb{Z}_p^n$ for some nonnegative integer n, the induced function $V \to M$ is locally analytic (i.e., locally represented by a convergent power series expansion). Let $C^i_{\mathrm{an}}(\Gamma, M) \subseteq C^i_{\mathrm{cont}}(\Gamma, M)$ be the space of analytic cochains.

Suppose now that M is an analytic Γ-module. Then by the proof of [8, Proposition V.2.3.6.3], $C^i_{\mathrm{an}}(\Gamma, M)$ is a subcomplex of $C^i_{\mathrm{cont}}(\Gamma, M)$; we thus obtain *analytic cohomology* groups $H^i_{\mathrm{an}}(\Gamma, M)$ and natural homomorphisms $H^i_{\mathrm{an}}(\Gamma, M) \to H^i_{\mathrm{cont}}(\Gamma, M)$.

Theorem 3.8 (Lazard)**.** *If M is an analytic Γ-module, then the inclusion $C^i_{\mathrm{an}}(\Gamma, M) \to C^i_{\mathrm{cont}}(\Gamma, M)$ is a quasi-isomorphism. That is, the continuous cohomology of M can be computed using analytic cochains.*

Proof. In the context of Example 3.6, this is the statement of [8, Théorème V.2.3.10]. However, the proof of this statement only uses the stronger hypothesis in the proof of [8, Proposition V.2.3.6.1], which we have built into the definition of an analytic Γ-module. The remainder of the proof of [8, Théorème V.2.3.10] thus carries over unchanged. $\square$

Remark 3.9. In considering Theorem 3.8, it may help to consider the first the case of 1-cocycles: every 1-cocycle is cohomologous to a crossed homomorphism, which is analytic because of how it is determined by its action on topological generators.

4 The HS property for some analytic group actions

We now establish our main result, which gives an analogue of the HS property for certain analytic group actions.

Theorem 4.1. *Let Γ be a profinite p-analytic group. Let H be a pro-p procyclic subgroup of Γ (i.e., it is isomorphic to $\mathbb{Z}_p$). Let M be a analytic Γ-module which is a Banach space over some nonarchimedean field of characteristic p with a nontrivial absolute value. (It is not necessary to require Γ to act on this field.) If $H^i_{\mathrm{cont}}(H, M) = 0$ for all $i \geq 0$, then $H^i_{\mathrm{cont}}(\Gamma, M) = 0$ for all $i \geq 0$.*

Proof. Let η be a topological generator of H. The vanishing of $H^0_{\mathrm{cont}}(H, M)$ and $H^1_{\mathrm{cont}}(H, M)$ means that $\eta - 1$ is a bijection on M; by the Banach open mapping theorem [2, §I.3.3, Théorème 1], $\eta - 1$ admits a bounded inverse. Since M is of characteristic p, for each nonnegative integer n the actions of $\eta^{p^n} - 1$ and $(\eta - 1)^{p^n}$ coincide; hence $\eta^{p^n} - 1$ also has a bounded inverse.

We next make some reductions. Recall that M has been assumed to be an analytic Γ-module. We may thus choose a pro-p-subgroup Γ_0 of Γ on which the logarithm map defines a bijection with $\mathbb{Z}_p^h$ for some h, such that for some $c_0 \in (0, 1)$ we have

$$|(\gamma - 1)y| \leq c_0 \, |y| \qquad (\gamma \in \Gamma_0, y \in M).$$

By the previous paragraph, we may also assume $\eta \in \Gamma_0$. Using Lemma 3.3, we may also assume $\Gamma = \Gamma_0$. By Theorem 3.8, to check that $H^i_{\mathrm{cont}}(\Gamma, M) = 0$ it suffices to check that $H^i_{\mathrm{an}}(\Gamma, M) = 0$.

Let Γ_n be the subgroup of Γ_0 which is the image of $p^n \mathbb{Z}_p^h$ under the exponential map. For c_0 as above, we have

$$|(\gamma - 1)(y)| \leq c_0^{p^n} \, |y| \qquad (n \geq 0, \gamma \in \Gamma_n, y \in M). \tag{4.1.1}$$

For $c \in (0, c_0]$, we say that a cochain $f : \Gamma^n \to M$ is *c-analytic* if there exists $d > 0$ such that

$$|f(\gamma_1, \dots, \gamma_n) - f(\gamma_1 \eta_1, \dots, \gamma_n \eta_n)| \leq d c^{p^{i_1 + \cdots + i_n}} \quad (\gamma_1, \dots, \gamma_n \in \Gamma; i_1, \dots, i_n \geq 0; \eta_j \in \Gamma_{i_j}). \tag{4.1.2}$$

Using the fact that M is of characteristic p, one may check that any analytic cochain in the sense of Lazard is c-analytic for some $c > 0$. This means that $C^n_{\mathrm{an}}(\Gamma, M)$ can be written as

the union of the subspaces $C^n_{\mathrm{an},c}(\Gamma, M)$ of c-analytic cochains over all $c \in (0, c_0]$. Moreover, using (4.1.1) we see that $C^n_{\mathrm{an},c}(\Gamma, M)$ is a subcomplex of $C^n_{\mathrm{an}}(\Gamma, M)$, so to prove the theorem it suffices to check the acyclicity of each $C^n_{\mathrm{an},c}(\Gamma, M)$.

From now on, fix $c \in (0, c_0]$. We define a norm on $C^n_{\mathrm{an},c}(\Gamma, M)$ assigning to each cochain f the minimum $d \geq 0$ for which (4.1.2) holds; note that $C^n_{\mathrm{an},c}(\Gamma, M)$ is complete with respect to this norm. For $m \geq 0$, we define a chain homotopy h_m on $C^\cdot_{\mathrm{an},c}(\Gamma, M)$ by the following formula: for $f_n \in C^n_{\mathrm{an},c}(\Gamma, M)$,

$$h_m(f_n)(\gamma_1, \dots, \gamma_{n-1}) = (\eta^{p^m} - 1)^{-1} \sum_{i=1}^n (-1)^{i-1} f_n(\gamma_1, \dots, \gamma_{i-1}, \eta^{p^m}, \gamma_i, \dots, \gamma_{n-1}).$$

We then compute that

$$\begin{aligned}
&(d \circ h_m + h_m \circ d - 1)(f_n)(\gamma_1, \dots, \gamma_n) \\
&= (\gamma_1(\eta^{p^m} - 1)^{-1} - (\eta^{p^m} - 1)^{-1}\gamma_1) \sum_{i=1}^n (-1)^{i-1} f_n(\gamma_2, \dots, \gamma_i, \eta^{p^m}, \gamma_{i+1}, \dots, \gamma_n) \\
&- \sum_{i=1}^n (\eta^{p^m} - 1)^{-1}(f_n(\gamma_1, \dots, \gamma_{i-1}, \eta^{p^m}\gamma_i, \gamma_{i+1}, \dots, \gamma_n) - f_n(\gamma_1, \dots, \gamma_{i-1}, \gamma_i\eta^{p^m}, \gamma_{i+1}, \dots, \gamma_n)).
\end{aligned}$$

To bound the right side of this equality, write

$$\gamma(\eta^{p^m} - 1)^{-1} - (\eta^{p^m} - 1)^{-1}\gamma = (\eta^{p^m} - 1)^{-1}(\eta^{p^m}\gamma)(1 - \gamma^{-1}\eta^{-p^m}\gamma\eta^{p^m})(\eta^{p^m} - 1)^{-1}.$$

Then note that if $\gamma_i \in \Gamma_j$, then $\eta^{p^m}\gamma_i$ and $\gamma_i\eta^{p^m}$ differ by an element of Γ_{m+j+1}. Finally, let $t > 0$ be the operator norm of the inverse of $\eta - 1$ on M; then $\eta^{p^m} - 1$ has an inverse of operator norm at most t^{p^m}. Fix $\epsilon \in (0, 1)$; for m sufficiently large, we have

$$\max\{t^{2p^m} c^{p^{2m}}, t^{p^m} c^{p^{m+1}}\} < 1 - \epsilon.$$

For such m, the map $d \circ h_m + h_m \circ d - 1$ acts on $C^n_{\mathrm{an},c}(\Gamma, M)$ with operator norm at most $1 - \epsilon$; consequently, there is an invertible map on $C_{\mathrm{an},c}(\Gamma, M)$ which is homotopic to zero. This proves the claim. $\square$

Note that Example 2.6 and Example 2.7 show that Theorem 4.1 cannot remain true if we drop the condition that H be pro-p. However, it does not resolve the following question.

Question 4.2. Does Theorem 4.1 remain true if we drop the condition that H be procyclic? This does not follow from Theorem 4.1 because the hypothesis of the theorem is not preserved upon replacing H with a subgroup (Remark 2.2).

5 Examples from p-adic Hodge theory

We conclude with some examples of Theorem 4.1 which are germane to p-adic Hodge theory.

Definition 5.1. For any ring R of characteristic p, let $\overline{\varphi}: R \to R$ denote the p-power Frobenius endomorphism.

Remark 5.2. We will frequently use the "Leibniz rule" for group actions, in the form of the identity

$$(\gamma - 1)(\overline{x}\overline{y}) = (\gamma - 1)(\overline{x})\overline{y} + \gamma(\overline{x})(\gamma - 1)(\overline{y}). \tag{5.2.1}$$

For instance, this holds if γ acts on a ring containing $\overline{x}$ and $\overline{y}$, or if it acts compatibly on a ring containing $\overline{x}$ and a module containing $\overline{y}$ (or vice versa).

Proposition 5.3. *Let F be a complete discretely valued field of characteristic p. Let R be an affinoid algebra over F. Let M be a finitely generated R-module. Let Γ be a profinite p-analytic group acting compatibly on F, R, M, and suppose that there is an open subgroup of Γ fixing the residue field of F. Then M is an analytic Γ-module.*

Proof. Let $\mathfrak{o}_F$ be the valuation subring of F. Let $\overline{\pi}$ be a uniformizer of $\mathfrak{o}_F$. By hypothesis, there exists an open subgroup Γ_0 on Γ fixing $\mathfrak{o}_F/(\overline{\pi})$. Then for any $\gamma \in \Gamma_0$ and any positive integer n, γ^{p^n} fixes $\mathfrak{o}_F/(\overline{\pi}^{n+1})$, so F itself is an analytic Γ-module.

By definition, R is a quotient of the Tate algebra $F\{T_1, \dots, T_n\}$ for some nonnegative integer n. Equip R with the quotient norm for some such presentation. Let $r_i \in R$ be the image of T_i. Since the action of Γ on R is continuous, for any $c > 0$ there exists an open subgroup Γ_0 of Γ such that

$$|(\gamma - 1)(f)| \leq \frac{c}{2} |f|, \qquad |(\gamma - 1)(r_i)| \leq \frac{c}{2}$$

for all $\gamma \in \Gamma_0$, $i \in \{1, \dots, n\}$, $f \in F$. Then for any $x \in R$, we can lift it to some $y = \sum_{i_1,\dots,i_n=0}^{\infty} y_{i_1,\dots,i_n} T_1^{i_1} \cdots T_n^{i_n} \in F\{T_1, \dots, T_n\}$ with $|y| \leq 2|x|$, and then observe that

$$\begin{aligned}
|(\gamma - 1)(x)| &\leq \max\{\left|(\gamma - 1)(y_{i_1,\dots,i_n} r_1^{i_1} \cdots r_n^{i_n})\right| : i_1, \dots, i_n \geq 0\} \\
&\leq \frac{c}{2} \max\{|y_{i_1,\dots,i_n}| : i_1, \dots, i_n \geq 0\} \qquad \text{(by (5.2.1))} \\
&= (c/2)\,|y| \leq c\,|x|.
\end{aligned}$$

It follows that the action of Γ on R is analytic.

Since R is noetherian, M may be viewed as a finite Banach module over R by [1, Proposition 3.7.3/3, Proposition 6.1.1/3]. By choosing topological generators for M as an R-module, we may repeat the argument of the previous paragraph to deduce that the action of Γ on M is analytic. $\square$

Example 5.4. The action of $\Gamma = \mathbb{Z}_p^\times$ on $F = \mathbb{F}_p((\overline{\pi}))$ via the substitution $\pi \mapsto (1+\pi)^\gamma - 1$ is analytic. By contrast, the induced action on the completion of the perfect closure of F is continuous but not analytic.

Now take $R = F$ and $M = \overline{\varphi}^{-1}(R)/R$. By Proposition 5.3, the action of Γ on M is analytic.

Put $\gamma = 1 + p^2 \in \Gamma$; this element generates the pro-p procyclic subgroup $H = 1 + p^2\mathbb{Z}_p$ of Γ. As an H-module, M splits as a direct sum $\bigoplus_{j=1}^{p-1}(1+\overline{\pi})^{j/p}F$. Choose $j \in \{1, \dots, p-1\}$ and put $\overline{y} = (1+\overline{\pi})^{j/p}$. We have

$$(\gamma - 1)(\overline{\pi}) = (\gamma - 1)(1 + \overline{\pi}) = ((1+\overline{\pi})^{p^2} - 1)(1 + \overline{\pi}).$$

Thus on one hand,

$$|(\gamma - 1)(\overline{x})| \leq |\overline{\pi}|^{p^2} |\overline{x}| \qquad (\overline{x} \in F);$$

on the other hand,

$$|(\gamma - 1)(\overline{y})| = |\overline{\pi}|^{p} |\overline{y}|,$$

and by (5.2.1), we see that for all $\overline{z} \in \overline{y}F$ we have

$$|(\gamma - 1)(\overline{z})| = |\overline{\pi}|^{p} |\overline{z}|.$$

In particular, $\gamma - 1$ is bijective on $\overline{y}F$ for each j, so $H^i_{\mathrm{cont}}(H, M) = 0$ for all $i \geq 0$. In this example, H is normal in Γ, so we may invoke Lemma 3.3 to deduce that $H^i_{\mathrm{cont}}(\Gamma, M) = 0$ for all $i \geq 0$. This calculation plays an essential role in the proof of the Cherbonnier-Colmez theorem described in [6].

This example generalizes as follows.

Example 5.5. Put $F = \mathbb{F}_p((\overline{\pi}))$ and $R = F\{\overline{t}_1, \dots, \overline{t}_d\}$ for some $d \geq 0$. The ring R admits a continuous action of $\Gamma = \mathbb{Z}_p^\times \rhd \mathbb{Z}_p^d$ in which $\gamma \in \mathbb{Z}_p^\times$ acts as in Example 5.4 fixing $\mathbb{Z}_p^d$, while for $j = 1, \dots, d$ an element γ_j in the j-th copy of $\mathbb{Z}_p$ sends $\overline{t}_j$ to $(1+\overline{\pi})^{\gamma_j}\overline{t}_j$ and fixes $\overline{\pi}$ and $\overline{t}_k$ for $k \neq j$. Put $M = \overline{\varphi}^{-1}(R)/R$. By Proposition 5.3, the actions of Γ on F, R, M are analytic.

Put $\Gamma_0 = (1 + p^2\mathbb{Z}_p) \rhd p\mathbb{Z}_p^d$. We then have a decomposition

$$M \cong \bigoplus (1+\overline{\pi})^{e_0/p}\overline{t}_1^{e_1/p} \cdots \overline{t}_d^{e_d/p} R \tag{5.5.1}$$

of R-modules and Γ_0-modules, in which $(e_0, \dots, e_d)$ runs over $\{0, \dots, p-1\}^{d+1} \setminus \{(0, \dots, 0)\}$.

Choose a tuple $(e_0, \dots, e_d) \neq (0, \dots, 0)$ and put $\overline{y} = (1+\overline{\pi})^{e_0/p}\overline{t}_1^{e_1/p} \cdots \overline{t}_d^{e_d/p}$. Suppose first that $e_j \neq 0$ for some $j > 0$. Let γ be the canonical generator of the j-th copy of $p\mathbb{Z}_p^d$. Then

$$|(\gamma - 1)(\overline{y})| = |\overline{\pi}|\, \overline{y};$$

on the other hand,

$$|(\gamma - 1)(\overline{x})| \leq |\overline{\pi}|^{p} |\overline{x}| \qquad (\overline{x} \in R),$$

so using (5.2.1) again we see that $\gamma - 1$ acts invertibly on $\overline{y}R$. By Lemma 3.3 we have $H^i_{\mathrm{cont}}(\Gamma_0, \overline{y}R) = 0$ for all $i \geq 0$.

Suppose next that $e_0 \neq 0$ but $e_1 = \cdots = e_d = 0$. Put $\gamma = 1 + p^2 \in \mathbb{Z}_p^\times$. As in Example 5.4, we see that $\gamma - 1$ acts invertibly on $\overline{y}R$. Since $\mathbb{Z}_p^\times$ is not normal in Γ, we must now apply Theorem 4.1 instead of Lemma 3.3 to deduce that $H^i_{\mathrm{cont}}(\Gamma_0, \overline{y}R) = 0$ for all $i \geq 0$.

Putting everything together, we deduce that $H^i_{\mathrm{cont}}(\Gamma_0, M) = 0$ for all $i \geq 0$. By Lemma 3.3 once more, we see that $H^i_{\mathrm{cont}}(\Gamma, M) = 0$ for all $i \geq 0$. This calculation plays an essential role in a generalization of the Cherbonnier-Colmez theorem described in [7].

Remark 5.6. Another class of examples to be considered in [7], based on Lubin-Tate towers, yields cases in which $\Gamma = \mathrm{GL}_d(\mathbb{Z}_p)$ and the vanishing of cohomology can again be checked using Theorem 4.1.

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